EXPLORING THE RESIDUAL QUEUE LENGTH EQUATION IN THE SHOCK WAVE MODEL

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Introduction

The shock wave theory was proposed by Lighthill and Whitham (1955) and Richards (1956). For a signalized intersection, this theory was developed to analyze the dynamics of queue formation along the cycles. In an oversaturated condition, part of the queue cannot be discharged at the end of a cycle. This generates residual queue for other cycles, causing delays or even blocking upstream ramps and intersections. Given this, finding the residual queue length is significant to the calculation of travel times and delays. Reviewing the literature, we found nine papers that have used the residual queue length equation. However, the equation has been presented in six different forms, and some of them are completely different from the others. Since the equation is based on a given geometrical proof, it should be identical throughout the literature, but this is not the case. Hence, in this paper we aim to investigate if the existing variations of the residual queue equation generate the same and correct values.

The shock wave formulation

Generally, the shock wave speed identifies a considerable, abrupt change in the average flow rate and density in two different regions, and is defined as below (*i* and *j* show two different regions):

$$SW_i = \frac{q_j - q_i}{k_j - k_i} \tag{1}$$

The details of the theory and the application of the method can be found in Stephanopoulos and Michalopoulos (1979). The formation of different shock waves in a signalized intersection are shown in Fig. 1 and their equations are summarized below:

$$SW_I = \frac{q}{k_a - k_j} \tag{2}$$

$$SW_R = \frac{S}{k_d - k_j} \tag{3}$$

$$SW_N = \frac{S - q}{k_d - k_a} \tag{4}$$

$$SW_A = \frac{q}{k_a} \tag{5}$$

q, and S are the average arrival and saturated flow rates (*veh/sec*), accordingly. k_a, k_d, and k_j denote arrival, discharge, and jam density (*veh/m*). The time to the maximum extent of the queue, the maximum extent of the queue, and the time to clear the maximum extent of the queue are given by the equations below:

$$t_m = \frac{q \cdot r\left(k_j - k_d\right)}{S\left(k_j - k_a\right) - q\left(k_j - k_d\right)} \tag{6}$$

$$X_m = -t_m \cdot SW_R = \frac{q \cdot r \cdot S}{S(k_j - k_a) - q(k_j - k_d)} \tag{7}$$

$$t_c = \frac{X_m}{SW_N} = \frac{q \cdot r \cdot S(k_d - k_a)}{(S - q)\left[S(k_j - k_a) - q(k_j - k_d)\right]} \tag{8}$$

Where r is the effective red time (*sec*), g is the effective green time (*sec*), and C is the cycle time (*sec*), t_m and t_c are in units of (*sec*) and X_m in (*m*).

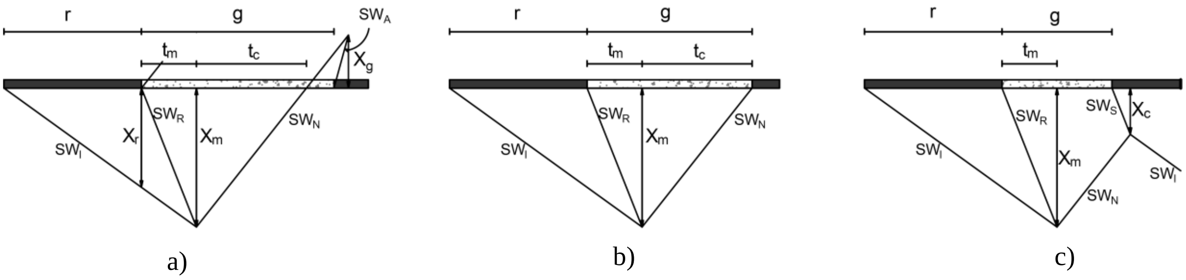

Figure 1. Shockwave analysis in three conditions: a) undersaturated b) saturated c) oversaturated

When all the vehicles that approached the intersection cannot pass the intersection in a cycle time, another shock wave propagates backward called SW_S:

$$SW_S = \frac{S}{k_{j(n)} - k_d} \tag{9}$$

The jam density in the next cycle is denoted by $k_{j(n)}$. If $k_{j(n)}$ is considered equal to the current cycle's density, SW_S will be the same as SW_R. The intersection of this wave with SW_N, specifies the residual queue length at the end of the cycle, which will be studied in the next section.

Investigating the correct residual queue length equation

We collected six different equations (equations 10-15) from nine publications that were presented to find the residual queue length in the oversaturated condition, which are listed below:

Stephanopoulos and Michalopoulos 1979	$X_c = X_m - (g - t_m + t_0) SW_N$ Where: $t_0 = \dfrac{X_m - (g - t_m) SW_N}{SW_N + SW_S}$	(10)
Dion et al. 2004	$X_c = \dfrac{C(q - S)}{k_j}$	(11)
Liu et al. 2009, Wu et al. 2010, Gettman et al. 2012	$X_c = \left(\dfrac{X_m}{SW_N} + t_m - g\right) \Big/ \left(\dfrac{1}{SW_N} + \dfrac{1}{SW_S}\right)$	(12)
Ponlathep 2010 & 2011	$X_c = \dfrac{SW_S}{(SW_S - SW_I)} \cdot \left(\dfrac{r \cdot SW_I \cdot SW_N + r \cdot SW_S \cdot SW_I - g \cdot SW_N \cdot SW_S + g \cdot SW_N \cdot SW_I}{SW_N + SW_S}\right)$	(13)
Ban et al. 2011	$X_c = t_1 \left(\dfrac{SW_S \cdot SW_N}{SW_S + SW_N}\right)$	(14)
Cetin 2012	$X_c = X_m - g\left(\dfrac{SW_S \cdot SW_N}{SW_S - SW_N}\right)$	(15)

All parameters in these equations are shown based on the equivalent parameters defined earlier in this paper. Simplifying these equations helped us discover that the equations are not the same. To investigate further, we used a numerical example given in Table 1 to compare the outputs of all equations, considering the parameters as S = 1800 (*veh/hr*) = 0.5 (*veh/sec*), k_j = 120 (*veh/km*) = 0.12 (*veh/m*), k_d = 60 (*veh/km*) = 0.06 (*veh/m*). This revealed that equations 10, 12, and 14 are equivalent (although t_1 has not been given as a specific equation; it is equal to the numerator of equation 11); the only issue in these equations is that the addition operator (+) before SW_S must be changed to subtraction (-), because SW_S is a negative number and its sign is opposite to SW_N, while the goal here is adding the abstract value of SW_S to SW_N. If we complete this change, the three equations will become equivalent to each other and to the correct equation 15 provided by Cetin (2012). We provide the correct simplified form of the residual queue length in accordance to these four equations (if the operator change is considered in the equations 10, 12, and 14) as below:

$$X_c = \frac{S(q \cdot C - g \cdot S)}{S(k_j - k_a) - q(k_j - k_d)} \tag{16}$$

Nevertheless, based on the numerical example provided in Table 1, the five equations (10, 12, 14, 15, and 16) show different values in two cases. First, in a saturated case, the equation proposed by Cetin

(2012) does not produce the expected value of zero like the other four equations. This is not appropriate if the residual queue length equation would be used as a criterion to differentiate undersaturated and oversaturated conditions, like in optimization models in Noaeen et al. (2016). Second, in the case of *v/c*=2.0, the equations 10, 15, and our presented equation 16 generate a number, while equations 12 and 14 give no number in this state.

We discuss the other two equations (11 and 13) separately because they are completely inconsistent with the other equations in the literature, both in the simplified form of the equation and the numerical example.

According to the numerical example in Table 1, equation 11 proposed by Dion et al (2004) generates negative numbers from *v/c* ratio of 0.1 to 1.9, and then in *v/c*=2.0 gives the residual queue length of zero in the oversaturated condition, while the correct trend is to generate the value of zero in the saturated condition, and positive values in *v/c* ratios greater than 1.0. Investigating this equation in terms of its parameters, we find three issues. First, the equation includes only the cycle time. This means changing green and red times has no impact on the traffic condition and the residual queue length, even if the green time would be zero. However, it is evident that when there is little or no green time in a cycle, we face an ever-growing queue, which leads to congestion and a reasonably over-saturated condition. Second, having only k_j in the denominator of the equation has an opposite impact on the calculations. Based on this equation, while jam density increases, the calculated residual queue length decreases. This relation is contrary to the fact that increase in jam density normally leads to the formation of more queues. Although there may be cases in which the increase in jam density does not increase the queue length, it is clear that it cannot always and directly force a decrease in the queue length. Finally, this equation is in contradiction with the equation of the degree of saturation in signalized intersections, which is given below:

$$v/c = q/S\left(g/C\right) \tag{17}$$

In each approach of a signalized intersection, *v/c* can be less than 1.0, exactly 1.0, or more than 1.0, where they are called undersaturated, saturated, and oversaturated, respectively. In the boundary of the saturation condition, *v/c* is 1.0. Therefore, in this condition we have:

$$v/c = q/S\left(g/C\right) = 1 \rightarrow qC - gS = 0 \tag{18}$$

qC-gS acts as a criteria to distinguish the three conditions mentioned above. When it is zero, it indicates the saturated condition. When it is less than zero, it reflects the undersaturated condition, and when it is positive it shows the oversaturated condition. The summation of t_m and t_c also provide the same criteria since:

$$t_m + t_c = \frac{q \cdot r}{S - q} \tag{19}$$

In the saturated condition t_m+t_c equals g and from the equation above we have:

$$q \cdot r + q \cdot g = g \cdot S$$

$$q \cdot C - g \cdot S = 0$$

As a result, the expression of *qC-gS* must appear in the numerator of the residual queue length equation to show the point of shifting from undersaturation to oversaturation, while *C(q-S)* in the numerator of equation 11 does not allow the equation to show this behaviour.

On the other hand, equation 13, proposed by Ponlathep (2010, 2011), in all cases produces negative values with a large difference in value compared to the other correct equations. It also becomes a long equation when it is simplified, not similar to the correct simplified form we presented in equation 16.

It is pertinent to note that the negative value obtained from X_c only means the intersection is in the undersaturated condition, and does not represent any distance, yet Ponlathep (2010, & 2011) considered it a distance after the stop-bar. This distance in the undersaturated condition that we show it with X_g in Fig. 1.a is obtained from the intersection of the waves SW_N and SW_A, not SW_S.

Table 1. Comparison of the residual queue length estimation from equations 9-15.

q (veh/hr)	*K_a* (veh/km)	*v/c*	Equations									
			(10)		(11)	(12)		(13)	(14)		(15)	(16)
			original	If (-)		original	If (-)		original	If (-)		
90	2	0.1	-13500	-117	-237	-13500	-117	-14256	-13500	-117	-117	-117
810	18	0.9	-166	-20	-137	-166	-20	-1006	-166	-20	-20	-20
900	20	1	0	0	-125	0	0	-857	0	0	1.42e-14	0
990	22	1.1	136	23	-112	136	23	-740	136	23	23	23
1710	38	1.9	710	540	-12	710	540	-609	710	540	540	540
1800	40	2	750	750	0	-	-	-750	-	-	750	750
1890	42	2.1	785	1100	12	785	1100	-1014	785	1100	1100	1100

Conclusion

In this paper, we studied the residual queue length equations for the shock wave delay model to determine which equation is correct and should be used in future calculations. We found six different equations in nine papers. Four equations (10, 12, 14, and 15) have been found correct, although some issues were discussed. Equation 10 is correct but was provided in two different equations. Equations 12 and 14 are correct if the sign before SW_S is changed to subtraction (-) as in equation 15. Equation 15 is also correct except that it does not generate the value of zero at the point of saturation. Two other equations (11, and 13) have been found as completely inconsistent, generating incorrect values. Then, a simplified form of the residual queue length (16) has been provided, which can be used easily in place of other equations in the literature; it is consistent with the equation of the degree of saturation, and also generates the correct values and signs in the three different saturation conditions.

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